Perfectoid Spaces: Properties, Examples, and Applications

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Last time Rahul talked about perfectoid fields, and in Student NT Joe talked about perfectoid spaces. This time we'll talk about perfectoid rings and spaces in more generality and untilts.

Fix always and forever a prime p.

1 Definitions and recollections

Definition 1. A complete Tate ring R is *perfectoid* if

1. R is uniform (i.e. the set of power-bounded elements $R^\circ \subset R$ is bounded),
2. there exists a topologically nilpotent unit $\varpi \in R$ such that $\varpi^p \mid p$ in R°, and
3. the p^{th} power Frobenius map
$$\Phi : R^\circ/\varpi \to R^\circ/\varpi^p$$
is an isomorphism.

A perfectoid field is a perfectoid ring which is also a non-archimedean field.

Definition 2. This definition is almost equivalent if we change condition 3 to

$$\Phi : R^\circ/p \to R^\circ/p \text{ is surjective}$$

The mod p version implies the mod ϖ version; the converse is true if $pR^\circ \subset R^\circ$ is closed, but probably not true in general. However, it sounds like coming up with an example of non-closed $pR^\circ \subset R^\circ$ is hard; in all examples we'll encounter it is closed, so we can treat this as an alternative definition.

In characteristic p, a perfectoid ring is a perfect uniform complete Tate ring; that's a lot of adjectives, but uniform and complete are more details, so one can think of a perfectoid ring as a perfect Tate ring.

In characteristic zero, one can think of perfectoid as a characteristic zero analogue of the perfect Tate ring condition. More intuition for this definition is that R is "highly ramified" at p; an element whose p^{th} power divides p and surjective Frobenius map are both about having lots of p^{th} roots. Roughly, to make something perfectoid, we can just add a bunch of p^{th} roots of things.

Example 3. Define $\mathbb{Q}_p^{\text{cycl}}$ to be the p-adic completion of $\mathbb{Q}_p(\zeta_{p^\infty})$. As discussed last time, this is a perfectoid field. Its subring of power-bounded elements is $\mathbb{Z}_p^{\text{cycl}}$, the completion of $\mathbb{Z}_p[\zeta_{p^\infty}]$; this is a perfectoid ring.

Example 4. $\mathbb{Q}_p^{\text{cycl}}\langle T\rangle$ is not perfectoid, as the Frobenius map

$$\overline{\mathbb{F}}_p\langle T\rangle \to \overline{\mathbb{F}}_p\langle T\rangle$$

is not surjective; but by adding p^{th} roots of T we can make it so. Define $\mathbb{Q}_p^{\text{cycl}}\langle T^{1/p^\infty}\rangle$ to be the result of p-adically completing $\mathbb{Z}_p^{\text{cycl}}[T^{1/p^\infty}]$ and then inverting p. Then $\mathbb{Q}_p^{\text{cycl}}\langle T^{1/p^\infty}\rangle$ is a perfectoid ring.

Example 5. An example of a perfectoid ring that is not an algebra over a perfectoid field is

$$\mathbb{Z}_p^{\text{cycl}}\langle (p/T)^{1/p^\infty}\rangle[1/T],$$

where for example we can take $\varpi = T^{1/p}$.

Everything we said about perfectoid fields and algebras carries over to the setting of perfetoid rings. The *tilt* of a perfectoid ring R is

$$R^\flat = \varprojlim_{x\mapsto x^p} R,$$

given the inverse limit topology and some weird addition law; this is again a perfectoid ring, and it has a (multiplicative but not additive) map $R^\flat \to \varprojlim R \to R$, written $f \mapsto f^\sharp$, by projecting onto the last coordinate; there exists a topologically nilpotent unit $\varpi \in R$ admitting a sequence of p^{th} roots, which give rise to a topologically nilpotent unit

$$\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \ldots) \in R^\flat,$$

and $(\varpi^\flat)^\sharp = \varpi$. This allows us to define the tilt in a slightly nicer way: first

$$R^{\flat\circ} = \varprojlim_{\Phi} R^\circ/\varpi,$$

this time as topological rings, and then $R^\flat = R^{\flat\circ}[\frac{1}{\varpi}]$.

The importance of tilting is that many things are "the same" for a perfectoid ring and its tilt. (Also many results for perfectoid fields generalize to perfectoid algebras).

Theorem 6. *Let R be a perfectoid ring.*

1. *Tilting $S \mapsto S^\flat$ gives an equivalence of categories*

$$\{\text{ perfectoid } R\text{-algebras }\} \xleftrightarrow{\sim} \{\text{ perfectoid } R^\flat\text{-algebras }\}.$$

2. *Every finite étale R-algebra is perfectoid.*

3. *Tilting gives an equivalence of categories*

$$\{\text{ finite étale } R\text{-algebras }\} \xleftrightarrow{\sim} \{\text{ finite étale } R^\flat\text{-algebras }\}.$$

One proves 3 by working on $\text{Spa}(R, R^\circ)$, using the equivalence over a perfectoid *field* to get an equivalence locally at every point, and then gluing. Then 2 is deduced by proving it in characteristic p and using 3 to get characteristic zero.

(We'll sketch a proof of 1 after talking about untilts.)

Lemma 7. *There is a bijection between rings of integral elements of R and $R^\flat$, given by $R^{\flat+} = \lim_{x \mapsto x^p} R^+$. Furthermore, $R^{\flat+}/\varpi^\flat = R^+/\varpi$.*

Theorem 8. *Let (R, R^+) be a perfectoid Huber pair with tilt $(R^\flat, R^{\flat+})$. Then there is a rational-subset-preserving homeomorphism*

$$\operatorname{Spa}(R, R^+) \cong \operatorname{Spa}(R^\flat, R^{\flat+})$$

where the image $x^\flat$ of x is given by $|f(x^\flat)| = |f^\sharp(x)|$ for $f \in R^\flat$.

Definition 9. A *perfectoid space* is an adic space X covered by $\operatorname{Spa}(R, R^+)$ with R perfectoid. By the previous result the tilts of an affinoid perfectoid cover will glue the same way the original affinoids do, so this extends to a tilting functor $X \to X^\flat$. If we fix a base, tilting gives an equivalence of categories between perfectoid spaces over blah and blah$^\flat$.

Quickly I want to mention a related concept that sometimes comes up. A Tate $\mathbb{Z}_p$-algebra R is *preperfectoid* if there is a perfectoid field K of characteristic zero such that $R \widehat{\otimes_{\mathbb{Z}_p}} \mathcal{O}_K$ is perfectoid. So preperfectoid is essentially an analogue of perfectoid over non-perfectoid fields. For example, $\mathbb{Q}_p\langle T^{1/p^\infty}\rangle$ is preperfectoid, but not perfectoid. (Unfortunately it is also possible to be perfectoid but not preperfectoid).

One can define related objects such as preperfectoid spaces in the usual way. This is important in contexts where we're not working over a perfectoid field. For example, infinite level Rapoport-Zink spaces (as in Scholze-Weinstein's paper) are actually preperfectoid rather than perfectoid.

Example 10. As we saw in Joe's talk, for K a perfectoid field the perfectoid closed disk over K is $\operatorname{Spa}(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle)$, and its tilt is simply $\operatorname{Spa}(K^\flat\langle T^{1/p^\infty}\rangle, \mathcal{O}_{K^\flat}\langle T^{1/p^\infty}\rangle)$, the perfectoid closed disk over $K^\flat$.

Example 11. We can make perfectoid versions of $\mathbb{A}^n$ and $\mathbb{P}^n$ by gluing perfectoid disks in a way similar to gluing adic disks to get $\mathbb{A}_K^{1,\mathrm{ad}}$ and $\mathbb{P}_K^{1,\mathrm{ad}}$. Since tilting glues, the fact that tilting the disk is simply tilting the field is true for these also:

$$(\mathbb{A}_K^{n,\mathrm{perf}})^\flat = \mathbb{A}_{K^\flat}^{n,\mathrm{perf}}, \qquad (\mathbb{P}_K^{n,\mathrm{perf}})^\flat = \mathbb{P}_{K^\flat}^{n,\mathrm{perf}}.$$

Example 12 (Special example for Bertie)**.** Let H be a p-divisible group over a perfect field of characteristic p. Define the *universal cover* of H to be $\widetilde{H} = \varprojlim_p H$, the inverse limit of H with respect to the multiplication by p map in the group law. This is a sheaf of $\mathbb{Q}_p$-vector spaces on the category of k-algebras.

Say H is connected. Then H is representable by $\operatorname{Spf} k[[T_1, \ldots, T_d]]$ and $\widetilde{H}$ is representable by $\operatorname{Spf} k[[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]]$ (where d is the dimension of H). Of course we can also analytify these and consider them as adic spaces. They are open disks of dimension d, or something like that.

Now $\widetilde{H}^{\mathrm{ad}}$ is almost perfectoid, but not quite because it is not analytic (i.e. $k[[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]]$ is not Tate). One way to fix this is to simply take the analytic locus: $\widetilde{H}^{\mathrm{ad}} \backslash 0$ is perfectoid. This is essentially one of the examples Joe did in his talk.

Alternatively, we can base change to a perfectoid field. Let K/k be a perfectoid field. Then

$$\begin{aligned} \widetilde{H}_K &= \text{ generic fiber of } \widetilde{H} \times_{\operatorname{Spec} k} \operatorname{Spf} \mathcal{O}_K \\ &(\overset{?}{=} \widetilde{H}^{\mathrm{ad}} \times_{\operatorname{Spa}(k,k)} \operatorname{Spa}(K, \mathcal{O}_K)) \end{aligned}$$

is a perfectoid space. In fact, it is a $\mathbb{Q}_p$-vector space object in the category of perfectoid spaces over K.

This thing is already in characteristic p so tilting does nothing. But if we want something similar in characteristic zero, we can untilt it. Suppose that K is a perfectoid field of characteristic zero whose residue field contains k. Then we can define $\widetilde{H}_K$ to be the perfectoid space over K whose tilt is $\widetilde{H}_{K^\flat}$.

H need not be representable by a formal scheme for this all to work; there is a functor $H \mapsto \widetilde{H}_K$ from the whole category of p-divisible groups over k to perfectoid spaces over K with $\mathbb{Q}_p$-vector space structure. (And in fact one can do this with any non-archimedean field whose residue field contains k, and get a preperfectoid space).

2 Untilts

Tilting is well behaved after we fix a base field or ring. How does tilting behave if we haven't fixed a base? More precisely, we know how to tilt perfectoid rings; how do we untilt them?

An *untilt* of a perfectoid Huber pair (R, R^+) is a perfectoid Huber pair $(R^\sharp, R^{\sharp+})$ together with an isomorphism $R^{\sharp\flat} \xrightarrow{\sim} R$ identifying $R^{\flat+}$ with R^+ via the correspondence given above. (Note that there's no restriction on the characteristic; R is an untilt of itself).

Untilting takes something in characteristic p and produces a similar thing in characteristic zero. It's unsurprising that this is related to our other favorite way to make characteristic zero from characteristic p, namely Witt vectors. The main fact we'll need is that the Witt ring $W(R)$ of (a perfect $\mathbb{F}_p$-algebra?) R is universal among p-adically complete rings A with continuous multiplicative map $R \to A$. That is, given a p-adically complete ring A with a continuous multiplicative map $R \to A$, there is a unique morphism of topological rings $W(R) \to A$ making the diagram commute.

$$\begin{tikzcd}
R \arrow[r] \arrow[dr] & W(R) \arrow[d] \\
& A
\end{tikzcd}$$

The continuous multiplicative map $R \to W(R)$ is denoted $x \mapsto [x]$, and the elements $[x]$ are called *Teichmüller representatives*.

Now, suppose we have a perfectoid ring Huber pair (R, R^+) with tilt $(R^\flat, R^{\flat+})$. What extra information do we need to include so that the tilt remembers where it came from? Recall that we have a continuous multiplicative map $R^{\flat+} \to R^+$, and R^+ is p-adically complete, so by the universal property we get a canonical map of topological rings $\theta : W(R^{\flat+}) \to R^+$. Furthermore, this map is surjective, because $R^{\flat+} \to R^+/p$ is surjective, so $W^{(}R^{\flat+}) \to R^+/p$ is surjective, and θ is surjective as well because both source and target are p-adically complete.

So in order to remember R^+ (and also $R = R^+[\varpi^{-1}]$) we only need to remember $I = \ker\theta \subset W(R^{\flat+})$. Now to classify untilts the question is: which ideals $I \subset W(R^{\flat+})$ can arise as $\ker\theta$?

Proposition 13. *An ideal $I \subset W(R^{\flat+})$ arises as $\ker\theta$ as above precisely when I is generated by a single element of the form $p - [\varpi]\alpha$, where $\varpi \in R^{\flat+}$ is a topologically nilpotent unit (unit in $R^\flat$ that is) and $\alpha \in W(R^{\flat+})$. Such ideals (and elements) are called* primitive of degree 1.

(Weinstein requires α to be a unit, is this right?)

Thus untilts of a perfectoid Huber pair (R, R^+) are classified by primitive ideals of degree 1 in $W(R^{\flat+})$.

Proposition 14. *Tilting gives an equivalence of categories between*

1. *perfectoid Huber pairs (R, R^+), and*

2. *triples $(R, R^+, \mathcal{J})$ where (R, R^+) is a perfectoid Huber pair in characteristic p and $\mathcal{J} \subset W(R^+)$ is a primitive ideal of degree 1.*

The equivalence is given by

$$
\begin{aligned}
(R, R^+) &\mapsto (R^\flat, R^{\flat +}, \ker\theta) \\
((W(R^+)/I)[[\varpi]^{-1}], W(R^+)/I) &\leftarrow\!\shortmid (R, R^+, I).
\end{aligned}
$$

All of this works also for untilts of perfectoid rings R (rather than Huber pairs). Untilts of R are classified by primitive ideals of degree 1 in $W(R^\circ)$, and the corresponding equivalence of categories as above is given by

$$
\begin{aligned}
R &\mapsto (R^\flat, \ker(\theta : W(R^{\flat\circ}) \to R^\circ)) \\
(W(R^\circ)/I)[[\varpi]^{-1}] &\leftarrow\!\shortmid (R, I).
\end{aligned}
$$

This also gives us the equivalence of categories over a fixed base. Let R be a perfectoid ring with tilt $R^\flat$.

$$\{ \text{ perfectoid } R\text{-algebras } \} \xleftrightarrow{\sim} \{ \text{ perfectoid } R^\flat\text{-algebras } \}$$

The forward functor we already know: $S \mapsto S^\flat$.

Now let S be a perfectoid $R^\flat$-algebra. Then S° is an algebra over $R^{\flat\circ}$, and $W(S^\circ)$ over $W(R^{\flat\circ})$. Thus using the fixed map

$$\theta : W(R^{\flat\circ}) \to R^\circ \to R$$

arising from the chosen untilt R of $R^\flat$, we can produce an untilt $W(S^\circ) \otimes_{W(R^{\flat\circ}),\theta} R$ of S. This is the backwards functor in the equivalence.

$$W(S^\circ) \otimes_{W(R^{\flat\circ}),\theta} R \leftarrow\!\shortmid S$$

3 The étale site

A morphism of perfectoid spaces $f : X \to Y$ is *finite étale* if for any affinoid $\mathrm{Spa}(B, B^+) \subset Y$ the preimage under f is an affinoid $\mathrm{Spa}(A, A^+)$ where A is a finite étale B-algebra and A^+ is the integral closure of (the image of) B^+ in A.

A morphism f is *étale* if it is (Zariski)-locally an open embedding followed by a finite étale map. That is, for all $x \in X$ there are opens U containing x and V containing $f(U)$ such that there exists a commuting diagram as follows.

$$
\begin{tikzcd}
U \arrow[r, "\text{open}"] \arrow[dr, "f"'] & W \arrow[d, "\text{finite étale}"] \\
& V
\end{tikzcd}
$$

An *étale cover* is a jointly surjective family of étale maps.

Proposition 15. *Étale maps are preserved by composition and pullback. They are also preserved by tilting: f is étale precisely when $f^\flat$ is.*

Thus étale covers define an étale site $X_{\text{ét}}$ on a perfectoid space X. Furthermore, tilting preserves the étale site: $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.

Proposition 16. *For any affinoid perfectoid space X and $i > 0$, the étale cohomology group $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ is almost zero.*